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ENDOMORPHISM RINGS OF B_2-GROUPS OF INFINITE RANK*

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ABSTRACT

B_2-groups are special (torsion-free) abelian Butler groups. The interest in this class of groups comes from representation theory. A particular functor, also called Butler functor, connects algebraic properties of the category of free abelian groups with (a few) distinguished subgroups with these Butler groups. This helps to understand Butler groups and caused lots of activities on Butler groups. Butler groups were originally defined for finite rank, however a homological connection discovered by Bican and Salce opened the investigation of Butler groups of infinite rank. Despite the fact that classifications of Butler groups are possible under restriction even for infinite rank (see a forthcoming paper by Files and Göbel [Mathematische Zeitschrift]), general structure theorems are impossible. This is supported by the following very special case of the Main Theorem of this paper, showing that any ring with a free additive group is an endomorphism ring of a Butler group. The result implies the existence of large indecomposable or of large superdecomposable Butler groups as well as the existence of counter-examples for Kaplansky's test problems.

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§1. Introduction

Thirty years ago M. C. R. Butler [10] introduced a class of torsion-free abelian groups that now carries his name. Recent years saw stormy new developments in the theory of these groups, much of which was motivated and fueled by interactions with representation theory of posets (see [1–5, 7–9, 11, 12, 35]). Butler groups of finite rank are defined as pure subgroups (or, equivalenty, torsion-free epimorphic images) of finite direct sums of subgroups of $\mathbb{Q}$. This definition cannot be extended in a meaningful way to groups of infinite rank. But then Bican and Salce [9] showed that Butler groups can be characterized by the equation $\mathrm{Bext}^1(B,T) = 0$ for all torsion groups T, i.e. each short balanced exact sequence $0 \to T \to X \to B \to 0$ splits. Since this condition makes sense for torsion-free groups B of any rank, one calls a torsion-free group B a Butler group (or B_1-group) if $\mathrm{Bext}^1(B,T) = 0$ for all torsion groups T. We refer to Fuchs [26, Vol. II] for basic properties of the functor $\mathrm{Bext}^1(\_,\_)$.

This homological definition of Butler groups (of any rank) opened up a whole new field of research in abelian group theory aimed at gaining an understanding of B_1-groups. Bican and Salce introduced a class of B_1-groups that are commonly called B_2-groups: A torsion-free abelian group B is called B_2-group if B is the union of a smooth ascending chain of pure subgroups $B_\alpha, \alpha < \lambda$, for some ordinal λ, i.e. $B = \bigcup_{\alpha<\lambda} B_\alpha$ such that each B_α is pure in B and $B_{\alpha+1} = B_\alpha + L_\alpha$ for some *finite* rank Butler group L_α. Each B_2-group is a B_1-group, cf. [9]. There are numerous partial results concerning the converse of this inclusion — see [18–20] for a sampling. The latest word in the matter might be [27]. It seems that set theoretic conditions play the important role (cf. [20]). In the present paper we want to investigate the richness of the class of B_2-groups. Despite the fact that some classes of Butler groups of finite rank are classified, this task seems to be hopeless for B_2-groups of infinite rank even if the typesets of the groups are tiny. First examples of B_2-groups are due to Fuchs and Metelli [28], who constructed large indecomposable as well as superdecomposable B_2-groups. Their construction is based on methods found in [16] or [25]. A different approach was utilized in [21] to construct *countable* B_2-groups. Here representations of modules with four distinguished submodules were the basic tool as developed in [31]. A generalization and unification of these results will be our main

THEOREM: *Let R be a ring with $1 \in R$ such that R^+ is a B_2-group and let $\lambda > |R|$ be a cardinal with $\lambda = \lambda^{\aleph_0}$. If R^+ is p-reduced for three distinct primes*

p, then there exist a B_2-group H such that $|H| = \lambda$ and $\operatorname{End} H \cong R$. *If R^+ is a free abelian group, then H can be chosen so as to have typeset $T_2 = \{\tau_1, \tau_2, \tau_0\}$ where τ_1, τ_2 are incomparable and $\tau_0 = \tau_1 \wedge \tau_2$.*

We have a topological version of this theorem, see (§4). These results have several applications, that are standard by now.

(a) For any infinite cardinal λ we find an indecomposable Butler group H of cardinality λ.

Here we use (4.2), a stronger version of the above theorem to deal with cardinals λ with $\lambda^{\aleph_0} \neq \lambda$.

(b) Let $(\Gamma, +)$ be an abelian semigroup. Then there exist B_2-groups $G_\gamma (\gamma \in \Gamma)$, such that for $\alpha, \beta \in \Gamma$, $G_\alpha \oplus G_\beta \cong G_{\alpha+\beta}$ and $G_\alpha \cong G_\beta$ if and only if $\alpha = \beta$. In particular, if $\Gamma = \mathbb{Z}/k\mathbb{Z}$, then there exists a B_2-group G such that $\bigoplus_m G \cong \bigoplus_n G$ if and only if $m \equiv n \bmod k$.

(c) If κ is any cardinal, then we can find a B_2-group G that is κ-superdecomposable, i.e. each summand $\neq 0$ of G decomposes into a direct sum of κ non-trivial summands

If κ is finite, this follows from the above theorem, and if κ is infinite, we employ the topological version.

(d) There exist pairwise non-isomorphic Butler groups G_0, G_1, G_2 such that $G_0 \oplus G_1 \cong G_0 \oplus G_2$.

(e) If λ is any infinite cardinal, in particular if $cf\lambda = \omega$, then we find a B_2-group G of cardinality λ such that G is not a direct sum of λ non-zero summands, but for each cardinal $\kappa < \lambda$ the group G decomposes into a direct sum of κ summands $\neq 0$.

Here we need the topological realization theorem which follows from [31], and not from the Black Box.

§2. Basic notions

The following definition is due to Hill [34]. A family $\mathcal{F}$ of pure subgroups of an abelian group A is an axiom-3 family if $0 \in \mathcal{F}$, $\sum \mathcal{X} \in \mathcal{F}$ for any $\mathcal{X} \subseteq \mathcal{F}$ and whenever $U \in \mathcal{F}$ and C is a countable subset of A, then there is $V \in \mathcal{F}$ with $U \cup C \subseteq V$ and V/U countable.

Let G be a torsion-free abelian group and A a pure subgroup of G. The subgroup A is called decent, cf. [4], if for any finite subset X of G there is a subgroup L of G such that L is a finite rank Butler group, $X \subseteq A + L$ and $A + L$ is pure in G. It was shown in [4] that B_2-groups are characterized as those torsion-free abelian groups that admit an axiom-3 family of decent subgroups; see also [27].

Let R be a ring with $1 \in R$. An R_2-module $\mathbf{F} = (F_0, F_1, F_2)$ is a free R-module F_0 together with two free submodules $F_1, F_2 \subseteq F_0$. The notion of axiom-3 families extends naturally to R_2-modules:

Let $\mathbf{F} = (F_0, F_1, F_2)$ be an R_2-module and $\mathcal{F}$ a family of R_2-submodules of $\mathcal{F}$. We call $\mathcal{F}$ an axiom-3 family if

(i) $(0,0,0) \in \mathcal{F}$.

(ii) *If $\mathcal{X} \subseteq \mathcal{F}$, then $\sum \mathcal{X} \in \mathcal{F}$, where $\sum \mathcal{X} = (S_0, S_1, S_2)$ and $S_i = \sum\{X_i\colon (X_0, X_1, X_2) \in \mathcal{X}\}$.*

(iii) *If $C \subseteq F_0$ is countable and $\mathbf{U} = (U_0, U_1, U_2) \in \mathcal{F}$, then there is $\mathbf{V} = (V_0, V_1, V_2) \in \mathcal{F}$ with $C \cup U_0 \subseteq V_0$ and V_i/U_i is a countably generated R-module for $i = 0, 1, 2$.*

For ease of notion, we say that $\mathbf{F} = (F_0, F_1, F_2)$ is in $\mathrm{Rep}_2(R)$ if F_i is free for $0 \le i \le 2$ and $F_1 \cap F_2 = 0$.

Let A be a torsion-free abelian group and p an integer. We say that A is p-reduced if $p^\omega A = 0$, where $p^\omega A = \bigcap_{n \in \omega} p^n A$ and we also say that A is Π-reduced for a set Π of integers if A is p-reduced for each $p \in \Pi$. Let $T_n = \{\tau_0, \tau_1, \ldots, \tau_n\}$ be a set of distinct types such that $\tau_0 = \tau_i \wedge \tau_j$ for $1 \le i \ne j \le n$. (Note that $\tau_1, \ldots, \tau_n$ are pairwise incomparable.) Let $B(T)$ be the class of all finite rank Butler groups with typeset contained in T, where T is any finite lattice of types. Any undefined notions may be found in [26]; in particular $t(A)$ denotes the torsion-subgroup of A and $D \sqsubset A$ denotes a summand of A.

§3. Basic lemmas

The following lemma appears in [3], for $R = \mathbb{Z}$. We will need a natural generalization:

Lemma 3.1: *Let Π be a set of primes and R be a ring with $1 \in R, 1 \ne 0$ such that R^+ is torsion-free and Π-reduced. If $\mathbf{F} = (F_0, F_1, F_2)$ is an R_2-module with $F_1 \cap F_2 = 0$ and F_i free for $i = 0, 1, 2$ such that $F_0/(F_1 \oplus F_2)$ is a Π-group, then we can find an axiom-3 family $\mathcal{F}$ of R_2 -submodules of $\mathbf{F}$ with the following properties:*

(a) *If* $\mathbf{X} = (X_0, X_1, X_2) \in \mathcal{F}$, *then* $X_1 \cap X_2 = 0$ *and* $X_i = F_i \cap X_0$ *for* $i = 0, 1, 2$.

(b) F_i/X_i *is a free R-module (in particular* $X_i \sqsubset F_i$ *) for* $i = 0, 1, 2$.

(c) $X_0/(X_1 \oplus X_2)$ *is a* Π*-group.*

The proof is an easy back and forth argument. Recall that F_i is a free R-modules for $i = 0, 1, 2$ and therefore we have bases which we keep fixed to introduce an axiom-3 family $\mathcal{F}_0$ of summands X of F_0 such that $F/X, F_i/X_i$ are free for $X_i = F_i \cap X$ and $i = 1, 2$. Using back and forth again it is easy to see that the axiom-3 family $\mathcal{F}_0$ induces a subfamily

$$\mathcal{F}' = \{D \in \mathcal{F}_0 | D \cap (F_1 \oplus F_2) = (D \cap F_1) \oplus (D \cap F_2)\}$$

which again is an axiom-3 family.

Finally let

$$\mathcal{F} = \{(X_0, X_1, X_2) | X_0 \in \mathcal{F}' \text{ and } X_i = X_0 \cap F_i\}.$$

This is an axiom-3 family satisfying (a), (b) and (c).

Next we specialize $\Pi = \{p\}$ (p a prime) and show how to use R_2 -modules to find groups with "many" decent subgroups.

LEMMA 3.2.: *Let R be a ring with R^+ torsion-free and let*

$$\mathbf{X} = (X_0, X_1, X_2) \subseteq \mathbf{F} = (F_0, F_1, F_2)$$

be R_2-modules with F_i free for $i = 0, 1, 2$. Let p be a prime and A_i subgroups of $\mathbb{Q}$ with $1/p \notin A_i$, $i = 1, 2$. Moreover assume that $F_0/(F_1 \oplus F_2)$ and $X_0/(X_1 \oplus X_2)$ are p-groups. If $B = X_0 + (A_1X_1 \oplus A_2X_2)$ and $H = F_0 + (A_1F_1 \oplus A_2F_2)$ then the following holds:

(I) *If X_i is pure in F_i for $i = 0, 1, 2$, then B is a pure R-submodule of H.*

(II) *If X_i is a summand of F_i for $i = 0, 1, 2$ and R^+ is a B_2-group, then B is decent in H.*

Proof: First we show

(a) $F_1 \oplus F_2 = F_0 \cap (A_1F_1 \oplus A_2F_2)$.

Since clearly $F_1 \oplus F_2 \subseteq F_0 \cap (A_1F_1 \oplus A_2F_2)$, we may consider

$$[F_0 \cap (A_1F_1 \oplus A_2F_2)]/(F_1 \oplus F_2) \subseteq [F_0/(F_1 \oplus F_2)] \cap [(A_1F_1 \oplus A_2F_2)/(F_1 \oplus F_2)].$$

However

$$(A_1F_1 \oplus A_2F_2)/F_1 \oplus F_2 \cong A_1F_1/F_1 \oplus A_2F_2/F_2 \cong \bigoplus_{j=1}^{2} \bigoplus_{\mathrm{rk}\, F_j} (A_j \otimes R/\mathbb{Z} \otimes R)$$

is a p'-group since $1/p \notin A_j$. On the other hand $F_0/F_1 \oplus F_2$ is a p-group. Thus $F_0 \cap (A_1F_1 \oplus A_2F_2) = F_1 \oplus F_2$.

(b) $X_0 \cap (F_1 \oplus F_2) = X_1 \oplus X_2$.

Clearly $X_1 \oplus X_2 \subseteq X_0 \cap (F_1 \oplus F_2)$ and we consider

$$[X_0 \cap (F_1 \oplus F_2)]/(X_1 \oplus X_2) \subseteq [X_0/(X_1 \oplus X_2)] \cap [(F_1 \oplus F_2)/(X_1 \oplus X_2)].$$

Now $X_0/(X_1 \oplus X_2)$ is a p-group by hypothesis and $(F_1 \oplus F_2)/(X_1 \oplus X_2) \cong (F_1/X_1) \oplus (F_2/X_2)$ is torsion-free since X_i is pure in F_i for $i = 1, 2$. Thus (b) follows.

Combining (a), (b) and $X_0 \subseteq F_0$ we have

$$X_0 \cap (A_1F_1 \oplus A_2F_2) = X_0 \cap [(A_1F_1 \oplus A_2F_2) \cap F_0] = X_0 \cap (F_1 \oplus F_2) = X_1 \oplus X_2.$$

Hence (b) can be improved to

(c) $X_0 \cap (A_1F_1 \oplus A_2F_2) = X_1 \oplus X_2$.

We will use the following trivial observation several times:

(∗) If G is an abelian group and U a torsion-free subgroup of G, then $t(G)$ is (naturally) isomorphic to a subgroup of G/U.

Recall $B = X_0 + A_1X_1 + A_2X_2, H = F_0 + A_1F_1 + A_2F_2$ and let $K = X_0 + A_1F_1 + A_2F_2$. Now we claim that

(d) $t(H/B)$ is a p-group.

Using (∗) for $U = K/B$ and $G = H/B$ it is enough to show that $G/U \cong H/K$ is a p-group and K/B is torsion-free:

By (a) we have $X_0 \subseteq F$ and by the modular law we have

$$\begin{aligned} H/K =& [(F_0 + (A_1F_1 \oplus A_2F_2)]/[X_0 + (A_1F_1 \oplus A_2F_2)] \\ \cong& F_0/[(X_0 + A_1F_1 + A_2F_2) \cap F_0] \\ =& F_0/[X_0 + (A_1F_1 + A_2F_2) \cap F_0] \\ =& F_0/[X_0 + (F_1 \oplus F_2)] \end{aligned}$$

which is a p-group because $F_0/(F_1 \oplus F_2)$ is a p-group by hypothesis. Moreover, using (c), $(A_1X_1 \oplus A_2X_2) \subseteq A_1F_1 \oplus A_2F_2$ and the modular law, we have

$$\begin{aligned} K/B =&[X_0 + (A_1F_1 \oplus A_2F_2)]/[X_0 + A_1X_1 + A_2X_2] \\ \cong&(A_1F_1 \oplus A_2F_2)/[X_0 \cap (A_1F_1 \oplus A_2F_2) + A_1X_1 + A_2X_2] \\ =&(A_1F_1 \oplus A_2F_2)/(A_1X_1 \oplus A_2X_2) \\ \cong&A_1(F_1/X_1) \oplus A_2(F_2/X_2) \end{aligned}$$

which is torsion-free since $A_i \subseteq \mathbb{Q}$ and F_i/X_i is torsion-free since R^+ is torsion-free and X_i is pure in F_i.

Claim (d) can be rephrased as

(e) B is p'-pure in H.

In order to show (I) it remains to be shown that

(f) B is p-pure in H.

If $ph = b$ is an equation with $b \in B$ and $h \in H$, then we can write

$$b = x_0 + b_1x_1 + b_2x_2, \quad h = f_0 + h_1f_1 + h_2f_2$$

with $b_i, h_i \in A_i$ and $x_i \in X_i, f_i \in F_i$.

Since $1/p \notin A_i$, $i = 1, 2$ the above equation becomes

$$pmh = p(mf_0 + mh_1f_1 + mh_2f_2) = mx_0 + mb_1x_1 + mb_2x_2 = mb$$

for some $m \in \mathbb{Z}$ with $p \not| m$ and $mb_i, mh_i \in \mathbb{Z}$.

The right-hand side is in X_0 as $mb_ix_i \in X_i \subseteq X_0$. However X_0 is pure in F_0 by hypothesis and we find $u \in X_0 \subseteq B$ with $pu = mx_0 + mb_1x_1 + mb_2x_2 = m(x_0 + b_1x_1 + b_2x_2) \in X_0$. Thus $pu \in B \cap mH = mB$ by (e) and p doesn't divide m. Thus $pu = mu'$ for some $u' \in B$. Since $p \not| m$ we have $u'' \in B$ with $u = mu''$. Thus $pu'' = b$ from $mpu'' = pu = mb$ and torsion-freeness.

Next we show (II) and to this end consider a finite subset C in H. We have to find a pure subgroup X of H with $C \subseteq X = B + L$, L a finite rank Butler group. Obviously

$$\begin{aligned} H/(F_1 \oplus F_2) &= [F_0 + (A_1F_1 \oplus A_2F_2)]/(F_1 \oplus F_2) \\ &\subseteq F_0/(F_1 \oplus F_2) + (A_1F_1 \oplus A_2F_2)/(F_1 \oplus F_2) \end{aligned}$$

where $F_0/(F_1 \oplus F_2)$ is a p-group and

$$(A_1F_1 \oplus A_2F_2)/(F_1 \oplus F_2) \cong A_1F_1/F_1 \oplus A_2F_2/F_2 \cong \bigoplus_{j=1}^{2} \bigoplus_{\mathrm{rk}\, F_j} ((A_jR)/R)$$

is a torsion p'-group. Thus

(g) $H/(F_1 \oplus F_2)$ is a torsion group.

Since C is finite by (g), there is an $n \in \mathbb{N}$ with $nC \subseteq F_1 \oplus F_2$ and we may assume $C \subseteq F_1 \oplus F_2$. Since we may replace $c \in C$ by its components in F_1 and F_2, we may assume $C \subseteq F_1 \cup F_2$. Recall that F_i/X_i is a free R-module and R^+ is a B_2-group. Now we may find an abelian group $Y_i = X_i \oplus B_i$ pure in F_i such that B_i has finite rank and $C \cap F_i \subseteq B_i$ for $i = 1, 2$. If $Y_0 = (Y_1 \oplus Y_2)_* \subseteq F_0$, then we claim

(h) $Y_0/(Y_1 \oplus Y_2)$ is a p-group.

Observe that $F_0/(F_1 \oplus F_2)$ is a p-group by hypothesis, containing

$$[Y_0 + (F_1 \oplus F_2)]/(F_1 \oplus F_2) \cong [Y_0 + (F_1 \oplus F_2)/(Y_1 \oplus Y_2)]/[(F_1 \oplus F_2)/(Y_1 \oplus Y_2)].$$

The denominator $(F_1 \oplus F_2)/(Y_1 \oplus Y_2) \cong (F_1/Y_1) \oplus (F_2/Y_2)$ is torsion-free because of the purity of Y_i in F_i, $i = 1, 2$. Employing $(*)$ we see that $Y_0/(Y_1 \oplus Y_2)$ must be a p-group.

Recall that $X_1 \oplus X_2 \subseteq Y_0$ and Y_0 is pure in F_0. Therefore $X_0 \subseteq Y_0$ since $X_0/(X_1 \oplus X_2)$ is a p-group. Since $X_0 \subseteq F_0$ and X_0 is a summand of F_0, the modular law implies that X_0 is a summand of Y_0 and the complement is isomorphic to

$$Y_0/X_0 = [(B_1 \oplus B_2 + X_0)/X_0]_* = (B_1 + B_2 + X_0)_*/X_0 = [(B_1 + B_2)_* + X_0]/X_0$$

which is a pure subgroup of the free R-module F_0/X_0. Now we use that R^+ is a B_2-group and that $B_1 \oplus B_2$ has finite rank. We infer that$(B_1 \oplus B_2)_* = L \subset \oplus R$ is a finite rank Butler group. If $X = Y_0 + (A_1 Y_1 \oplus A_2 Y_2)$, then X satisfies the assumptions on B in (I) and X is pure in H. On the other hand $Y_0 = X_0 + L$ by the previous remarks and $X_0 \subseteq B \subseteq X$ allows us to replace the definition of X by $X = L + X_0 + (A_1 Y_1 \oplus A_2 Y_2) = B + (L + A_1 B_1 + A_2 B_2)$ with $L + (A_1 B_1 \oplus A_2 B_2)$ as the desired Butler group of finite rank. Also $C \subseteq X$ by design and B is decent in H.

§4. Butler groups

Our construction of B_2-groups with prescribed endomorphism ring R will be based on a realization theorem of R as endomorphism ring of some more general cotorsion-free group G. In principle there are two different realization theorems for R available. The first one puts less strain on R, but requires a cardinal jump $|G| > |R|$ and even more due to a construction using Shelah's Black Box or some similar combinatorical argument — see [17] and [22]. The second one requires additional properties of R, but we can get away without cardinal trouble. Each type of realization theorem has its advantages regarding applications. We will provide both of them and apply them to $\mathrm{Rep}_2(R)$. In the process we will be changing categories several times:

$$\mathrm{Rep}_4(R) \to \text{ abelian groups } \to \mathrm{Rep}_2(R) \to B_2\text{-groups},$$

where $\mathrm{Rep}_4(R)$ denotes the category of free R-modules with four distinguished submodules.

PROPOSITION 4.1:

(i) *If R a cotorsion-free ring and $\lambda = \lambda^{\aleph_0}$ is a cardinal $> |R|$, then there exists a cotorsion-free group G with* $\mathrm{End}\, G \cong R$ *and* $|G| = \lambda$.

(ii) *Let R be a torsion-free ring such that R is p-reduced for a prime p and that $\widehat{R}$, the p-adic closure of R, has transcendence degree at least four over R. If λ is any cardinal $\geq |R|$, then we can find a torsion-free, p-reduced abelian group G with* $\mathrm{End}\, G \cong R$ *and* $|G| = \lambda$.

(iii) *If R is a ring of cardinality $< 2^{\aleph_0}$ such that R^+ is torsion-free of cardinality $\leq \lambda$ and p-reduced for some prime p with $\aleph_0 \leq \lambda \leq 2^{\aleph_0}$, then there exists a cotorsion-free, p-reduced abelian group G with* $\mathrm{End}\, G = R$ *and* $|G| = \lambda$.

In each of the cases (i), (ii), *and* (iii) *the group G is an R-module and there are free R-modules $F_i, i = 0, 1$, such that $0 \to F_0 \to F_1 \to G \to 0$ is a short exact sequence of R-modules. Moreover, if R is p-reduced for some prime p, then G is p-reduced, and if $\mathbb{Z}_{(p)}$ is the ring of integers localized at p, then* $\mathrm{Hom}(G, G \otimes \mathbb{Z}_{(p)}) = R \otimes \mathbb{Z}_{(p)}$ *and G contains a free R-module L of rank λ such that G/L is a p-group.*

Proof: (i) is a main result in [17] and (ii) follows from [25]. In [25] the "co-maximal" argument must be replaced by a transcendental argument, as in [32]. The existence of a free resolution follows in case (i) from the construction of G: The cotorsion-free group G is obtained as a union of a smooth chain of R-submodules G_α, $\alpha < \lambda^*$, an ordinal, such that $G_{\alpha+1} = < G_\alpha, y_n : n \in \omega >$ with $py_{n+1} - y_n = a_n \in G_\alpha$. By transfinite induction, there is a short exact sequence of free R-modules $0 \to K \to F_\alpha \overset{\pi_\alpha}{\to} G_\alpha \to 0$. Let $\{y_n^*: n \in \omega\}$ be free generators, set $F_{\alpha+1} = F_\alpha \bigoplus_{n\in\omega} y_n^* R$ and extend the map $\pi_\alpha: F_\alpha \to G_\alpha \to 0$ by sending y_n^*

onto y_n. Then $K_{\alpha+1} = K_\alpha \oplus \bigoplus_{n\in\omega}(py^*_{n+1} - y^*_n - \pi_\alpha^{-1}(a_n))R$ where $\pi_\alpha^{-1}(a_n)$ is a fixed preimage of a_n. One easily verifies that

$$0 \to K_{\alpha+1} \to F_{\alpha+1} \to G_{\alpha+1} \to 0$$

is short exact, and $K_\alpha \sqsubset K_{\alpha+1}, F_\alpha \sqsubset F_{\alpha+1}$ implies that $K = \bigcup_{\alpha<\lambda^*} K_\alpha$ and $F = \bigcup_{\alpha<\lambda^*} F_\alpha$ are free R-modules.

Similarly, case (ii) requires a short inspection of the generators used in [31]. The additional properties about $\mathrm{Hom}(G, G \otimes \mathbb{Z}_{(p)})$ and the existence of L are easily established. The first one requires a slight modification of the constructions and the latter is obvious from the construction. For instance in case (ii) the module L is just the free module F_0, where $\mathrm{End}(F_0, F_1, \ldots, F_4) = R$ in $\mathrm{Rep}_4(R)$, and in case (i) L is the free R-submodule generated by "B" and all the "y_α", $\alpha < \lambda^*$ in [17].

Inspection of the proof in [13] or [30] shows that (iii) holds.

LEMMA 4.2: *Let R be a ring with $1 \in R$ such that R^+ is a B_2-group and $\mathbf{F} = (F_0, F_1, F_2) \in \mathrm{Rep}_2(R)$ such that $F_0/(F_1 + F_2)$ a p-group and $A_i \subseteq \mathbb{Q}$ with $\frac{1}{p} \notin A_i$ for $i = 1, 2$. Then $F_R = F_0 + A_1F_1 + A_2F_2$ is a B_2-group.*

Proof: By the definition of B_2-groups, we must find an ascending smooth chain $\mathcal{F}^*$ of pure and decent subgroups of F_R with countable quotients of successive members and union equal to F_R. First we apply Lemma 3.1 for $\Pi = \{p\}$ and find an axiom-3 family $\mathcal{F}$ of R_2-submodules $\mathbf{X} = (X_0, X_1, X_2) \in \mathcal{F}$ of $\mathbf{F}$ such that $X_i = F_i \cap X_0$ and F_i/X_i is free for $i = 0, 1, 2$.

We may assume that $\mathbf{F} = \bigcup_{\alpha<\lambda^*} \mathbf{X}^{(\alpha)}, \mathbf{X}^{(\alpha)} = (X_0^{(\alpha)}, X_1^{(\alpha)}, X_2^{(\alpha)})$ with $\mathbf{X}^{(\alpha)} \in \mathcal{F}$ and $\mathbf{X}^{(\alpha+1)}/\mathbf{X}^{(\alpha)}$ free of countable rank for all $\alpha < \lambda^*$. We will refine the chain $\{\mathbf{X}^{(\alpha)}\colon \alpha < \lambda^*\}$ to construct the desired chain for F_R. Since R^+ is a B_2-group, we can write $R = \bigcup_{\alpha<\mu} R_\alpha$ where the R_α's are pure and decent subgroups of R with $R_{\alpha+1}/R_\alpha$ of finite rank.

First, let $X_R^{(\alpha)} = X_0^{(\alpha)} + A_1X_1^{(\alpha)} + A_2X_2^{(\alpha)} \in \mathcal{F}^*$ for $\alpha < \lambda^*$. By Lemma 3.2, $X_R^{(\alpha)}$ is pure and decent in F_R and $X_i^{(\alpha)} \oplus \bigoplus_{n<\omega} c_n^{(\alpha,i)}R = X_i^{(\alpha+1)}$ since $\mathbf{X}^{(\alpha)} \in \mathcal{F}$ implies $X_i^{(\alpha)} \sqsubset X_i^{(\alpha+1)}$ for $i = 0, 1, 2$.

Now let

$$X_{R_\beta} = X_R \oplus ((\bigoplus_{n<\omega} c_n^{(\alpha,0)}R_\beta) + (\bigoplus_{i=1}^{2}\bigoplus_{n<\omega} A_ic_i^{(\alpha,n)}R_\beta)).$$

By Lemma 3.2 the R-module X_{R_α} is pure and decent in $X_{R_{\alpha+1}}$ and hence in F_R. Therefore $X_R^{(\alpha+1)} = \bigcup_{\beta<\mu} X_{R_\beta}$ and $X_{R_{\beta+1}}/X_{R_\beta}$ has finite rank. Thus the chain $\{X_{R_\beta}\colon \beta < \mu\}$ is a chain to be put between $X_R^{(\alpha)}$ and $X_R^{(\alpha+1)}$ to witness that F_R is a B_2-group.

MAIN THEOREM 4.3: *Let R be a ring, $1 \in R$, such that R^+ is a B_2-group. Suppose that there are three distinct primes $p = p_0, p_1, p_2$ such that R^+ is p_i-reduced for $i = 0, 1, 2$. If λ is an infinite cardinal and $\lambda = |R| \geq 2^{\aleph_0}$, or $\mathrm{cf}\lambda = \omega$ and $\lambda \geq 2^{\aleph_0}$, we also assume*

(∗) *The p-adic closure of R has transcendence degree at least four over R. Then there exists a B_2-group H with $\mathrm{End}\, H = R$ and $|H| = \lambda$. Let τ_i be the type of $\mathbb{Q}^{(p_i)} = \{z/p_i^n\colon z \in \mathbb{Z}, n < \omega\}$ for i = 1,2 and $\tau_0 = \tau_1 \wedge \tau_2 = 0$. If R^+ is free, then H is a B_2-group with typeset $\{\tau_0, \tau_1, \tau_2\}$.*

Remark: Observe that a suitable choice of λ, e.g. $\lambda = \lambda^{\aleph_0} > |R|$, makes (∗) empty. However the case $\lambda = |R|$ will be of particular interest in our applications (e) in the introduction. The cardinal conditions allow us to apply three different types of realization theorems as summarized in (4.1).

Proof: Recall that Proposition 4.1 is applicable for any choice of λ and we have a free resolution $0 \to F_1 \to F_0 \to G \to 0$ with free R-modules F_i of rank λ and F_1 a submodule of F_0. Let $F_3 \subseteq F_0$ be a free R-module with $F_1 \cap F_3 = 0$ and $(F_3 \oplus F_1)/F_1 \cong L \subseteq G$ as in Proposition 4.1.

Let $F_3 = \bigoplus_{i<\kappa} \bigoplus_{n<\omega} e_{in} R$ and define $F_2 = \bigoplus_{i<\kappa} \bigoplus_{n\in\omega} (e_i - p^n e_{in})R$ where $F_1 = \bigoplus_{i<\kappa} e_i R$. Recall that we may assume that F_3 and F_1 have the same infinite rank. Clearly

$$F_1 \cap (F_2)_{*p} = 0, \tag{1}$$

where $(...)_{*p}$ denotes the p-purification of $(...)$, as follows by our choice of F_2 and $F_1 \cap F_3 = 0$. It is also clear that

$$F_1 \text{ is pure in } F_0, \tag{2}$$

because G is torsion-free. Moreover $F_0/(F_1 \oplus F_2)$ is an epimorphic image of G/L and hence a p-group by Proposition 4.1. We have

$$F_0/(F_1 \oplus F_2) \text{ is a } p\text{-group}, \tag{3}$$

and claim

$$\mathrm{Hom}(F_0/F_2, G \otimes \mathbb{Z}_{(p)}) = 0. \tag{4}$$

Let $\varphi\colon F_0 \to G \otimes \mathbb{Z}_{(p)}$ be such a homomorphism with $F_2\varphi = 0$. Then $e_i\varphi = p^n(e_{in})\varphi$ for all $i < \kappa, n < \omega$. Thus $e_i\varphi \in p^\omega(G \otimes \mathbb{Z}_{(p)})$. But $F_0/(F_1 \oplus F_2)$ is a p-group and $G \otimes \mathbb{Z}_{(p)}$ is torsion-free. Hence $\varphi = 0$. Now let $\mathbf{F} = (F_0, F_1, F_2) \in \mathrm{Rep}_2(R)$. Next we show

$$\mathrm{End}\,\mathbf{F} = R. \tag{5}$$

This also follows from [24], but we include the short proof for the convenience of the reader. If $\varphi \in \mathrm{End}\,\mathbf{F}$, then φ induces $\varphi'\colon F_0/F_1 \to F_0/F_1 \cong G$ and by the choice of G we have $\varphi' = r \in R$. Thus $\varphi - r\colon F_0 \to F_0$ maps F_0 into F_1. In particular, $F_2(\varphi - r) \subseteq F_2 \cap F_1 = 0$ by (i) and $\varphi - r$ induces a map from $F_0/F_1 + F_2$, a p-group, into F_0, a torsion-free group. This shows that $\varphi - r = 0$. We are finally ready to define our B_2-group. Let p_1, p_2 be the other two primes given in the hypothesis, let $A_i = \mathbb{Q}^{(p_i)}$, and define $H = F_0 + (A_1F_1 \oplus A_2F_2)$, a B_2-group by Lemma 4.1. Let τ_i denote the type A_i and $\tau_0 = \tau_1 \wedge \tau_2 = 0$. We will show that

(6) $H(\tau_1) = A_1F_1$ and $H(\tau_2) = (A_2F_2)_{*p}$;

(7) $\mathrm{End}\,H = R$; and

(8) if R is homogeneous of type 0, then the typeset of H is $\{\tau_0, \tau_1, \tau_2\}$.

To show (6) first observe that trivially $A_iF_i \subseteq H(\tau_i)$. Let $h \in H(\tau_i)$. Then $p^kh \in A_1F \oplus A_2F_2$ for some $k \in \mathbb{N}$ and since $A_1F_1 \oplus A_2F_2$ is p'-pure in H (since $F_0/F_1 \oplus F_2$ is a p-group) we have $|p^kh|_q^H = |p^kh|_q^{(A_1F_1 \oplus A_2F_2)}$ for all primes $q \neq p$. This shows that $p^kh \in A_iF_i$ and $h \in (A_2F_2)_{*p}$. Recall that F_1 is pure in F_0, $p \neq p_i$ for $i = 1, 2$ and $1/p \notin A_i$ for $i = 1, 2$. This implies that A_1F_1 is p-pure in H, and hence $h \in A_1F_1$ follows and (6) holds.

To show that (7) holds, consider $\varphi \in \mathrm{End}(H)$. Since $H(\tau_i)$, $i = 1, 2$, is fully invariant, we infer $H(\tau_1)\varphi = (A_1F_1)\varphi \subseteq A_iF_i$ for $i = 1, 2$. Note that $H/A_1F_1 = (F_0 + A_1F_1 + A_2F_2)/A_1F_1 \cong (F_0 + A_2F_2)/[A_1F_1 \cap (F_0 + A_2F_2)]$. In the first step of the proof of Lemma 3.2, we showed that $F_0 \cap (A_1F_1 + A_2F_2) = F_1 + F_2$ and we infer $H/A_1F_1 \cong (F_0 + A_2F_2)/F_1 = F_0/F_1 + (A_2F_2 \oplus F_1)/F_1 \subseteq G + A_2L \subseteq G \otimes \mathbb{Z}_{(p)}$ since $p \neq p_2$.

Consider $\varphi' = \varphi \,|F_0/F_1$. Note that F_0/F_1 is isomorphic to a subgroup of H/A_1F_1 and $\varphi' \in \mathrm{Hom}(G, G \otimes \mathbb{Z}_{(p)}) = R \otimes \mathbb{Z}_{(p)}$ by choice of G. Thus $\varphi' = r/m, m \in \mathbb{N}$ is relatively prime to p.

Let $\psi = \varphi m - r$. Then $\psi \in \mathrm{End}(H)$ and $H\psi \subseteq A_1F_1$. In particular $A_2F_2\psi \subseteq (A_2F_2)_{*p} \cap A_1F_1 = 0$ by (6). Thus ψ induces a map ψ': $F_0 \to A_1F_1$ with $\psi \,|F_2 = 0$. Since A_1F_1 is a free module over A_1R we may view A_1F_1 as a subgroup of $G \otimes \mathbb{Z}_{(p)}$, and ψ' induces a map $F_0/F_2 \to G \otimes \mathbb{Z}_{(p)}$. Now (4) implies $\psi' = 0$. We infer $\psi = 0$ and $\varphi = r/m$ since H/F_0 is torsion. Let e be an m-pure element in the R-module $A_1F_1 \oplus A_2F_2$. Note that e is m-pure if units are the only factors of m which divide e. Then $e\varphi = e\frac{r}{m} \in eR$ and $m = 1$ follows. This shows that (7) holds.

Now let us assume that R^+ is homogeneous of type $\tau_0 = 0$, i.e. R^+ is free, and pick $h \in H$. Then there is $\mathbf{X} = (X_0, X_1, X_2) \subseteq \mathbf{F} = (F_0, F_1, F_2)$ such that $mh \in X_0$ for some $m \in \mathbb{N}$ and $X_i \sqsubset F_i$ for $i = 0, 1, 2$. By Lemma 3.2 the R-module $B = X_0 + A_1X_1 + A_2X_2$ is pure in H and we may compute the type of h inside B. Since R^+ is free, there are free pure subgroups $Y_i \subseteq X_i$ such that $h \in Y_0 + (A_1Y_1 \oplus A_2Y_2) = B^*$, B^* is pure in B, and B^* is an almost completely decomposable group of finite rank. This shows that h is of type τ_i for some $i \in \{0, 1, 2\}$.

We want to conclude this paper by giving a "topological version" of Theorem 4.3. For ease of notation, we introduce the following

Definition 4.4: Let R be a ring with $1 \in R$: We say that R admits a B_2-topology if there is a topology τ on R such that

(1) τ is Hausdorff and R is complete in τ.
(2) τ is a linear topology induced by a family $\mathcal{T}$ of right ideals of R such that $(R/I)^+$ is a B_2-group for each $I \in \mathcal{T}$.
(3) There exist three distinct primes p_i such that $(R/I)^+$ is p_i-reduced for all $I \in \mathcal{T}$ and $0 \leq i \leq 2$.

With the help of this definition, we are now able to state our final

THEOREM 4.5: *Let R be a ring with a B_2-topology, and $T_2 = \{\tau_0, \tau_1, \tau_2\}$ be as above. Then there exists a B_2-group H such that* End H, *endowed with the finite topology, is topologically and algebraically isomorphic to R. If R/I is homogeneous of type τ_0 for each $I \in \mathcal{T}$, then T_2 is the typeset of H.*

Proof: We call an R-module of the form $R/I (I \in \mathcal{T})$ $\mathcal{T}$-cyclic. A direct sum of

$\mathcal{T}$-cyclic modules is called $\sum$-$\mathcal{T}$-cyclic: We now have to run through the whole paper — which we will leave to the reader — substituting "free R-modules" by $\sum$-$\mathcal{T}$-cyclic modules.

Moreover, we have to employ — for case (i) in Proposition 4.1 — the cotorsion-free topological version in [17]: There exists a cotorsion-free abelian group G such that $\operatorname{End} G$, with its finite topology, is isomorphic to R as topological rings. Here G is a union of a smooth chain $G = \bigcup_{\alpha<\lambda*} G_\alpha$ such that $G_{\alpha+1} = \langle G_\alpha, y_n^{(\alpha)} : n < \omega\rangle$ such that $py_{n+1}^{(\alpha)} - y_n^{(\alpha)} = a_n \in G_\alpha$ and for all $n < \omega$, $\operatorname{Ann}_R(y_n^{(\alpha)}) = I_\alpha \in \mathcal{T}$. Then (like in the proof of Proposition 4.1) there exists a $\sum$-$\mathcal{T}$-cyclic submodule L in G, and hence there are $\sum$-$\mathcal{T}$-cyclic modules F_0, F_1 such that $0 \to F_0 \to F_1 \to G \to 0$ is short exact.

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